Warwick Mathematics Exchange

MA134

Geometry & Motion

2022, August 30th

Desync, aka The Big Ree

Contents

Introduction

In *Geometry & Motion*, we mainly study functions of different sized real inputs and outputs, and vector fields. The title is somewhat misleading in this regard, in that very little geometry or mechanics is done. The vast majority of techniques developed in this module are required for further study of calculus, in particular, MA259 *Multivariable Calculus*, which is core for maths students.

This module is very computational in nature, with many questions simply asking you to perform relatively simple integral calculations or transformations of coordinate systems. This computational aspect is shared with MA133 *Differential Equations.*

This document is intended to broadly cover all the topics within the Geometry & Motion module. Basic integration skills from core A-Levels is assumed. Knowledge of arc length and surface area integrals (from Edexcel FP2 o.e.) is not assumed, but prior experience is certainly helpful.

This document is not designed to be a replacement for lecture notes - much of the content is covered in a different order than is taught in the course, so it is not recommended to learn the module from these notes unless you are familiar with most of the content already.

Due to the computational nature of this module, this document mainly consists of a checklist of how to solve different questions, with not much in the way of theory. A short section of additional techniques has been included at the end. For a more extensive set of additional techniques, please see my MA133 *Differential Equations* guide.

Disclaimer: This document was made by a first year student with a severe disinterest in calculus. Furthermore, this module was also my lowest scoring maths module. I make *absolutely no guarantee* that this document is complete nor without error. In particular, any content covered exclusively in lectures (if any) will not be recorded here. Additionally, this document was written at the end of the 2022 academic year, so any changes in the course since then may not be accurately reflected.

Notes on formatting

New terminology will be introduced in *italics* when used for the first time. Named theorems will also be introduced in *italics*. Important points will be **bold**. Common mistakes will be underlined. The latter two classifications are under my interpretation. YMMV.

Content not taught in the course will be outlined in the margins like this. Anything outlined like this is not examinable, but has been included as it may be helpful to know alternative methods to solve problems.

The table of contents above, and any inline references are all hyperlinked for your convenience.

Scalars are written in lowercase italics, c, or using greek letters, χ.

Vectors are written in lowercase bold, $\mathbf{v}$, or rarely overlined, $\overleftrightarrow{v}$, where more contrast or clarity is required.

Functions are written as whatever their outputs are, i.e., any function which returns a vector will be written in bold regardless of the domain.

History

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Authors

This document was written by R.J. Kit L., a maths student. I am not otherwise affiliated with the university, and cannot help you with related matters.

Please send me a PM on Discord @Desync#6290, a message in the WMX server, or an email to Warwick.Mathematics.Exchange@gmail.com for any corrections. (If this document somehow manages to persist for more than a few years, these contact details might be out of date, depending on the maintainers. Please check the most recently updated version you can find.)

If you found this guide helpful and want to support me, you can buy me a coffee!

(Direct link for if hyperlinks are not supported on your device/reader: ko-fi.com/desync.)

*Storing dates in big-endian format is clearly the superior option, as sorting dates lexicographically will also sort dates chronologically, which is a property that little and middle-endian date formats do not share. See ISO-8601 for more details. This footnote was made by the computer science gang.

1 Curves & Parametrisation

A function, $\mathbf{r}$, is *vector-valued* if it maps a number, $t \in \mathbb{R}$ to a vector $\mathbf{r}(t) \in \mathbb{R}^n, n > 1$.

If $n = 1$, then the function is instead *scalar-valued*. One way to represent a vector-valued function is to write it in terms of its scalar-valued components, such as $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where x, y and z are scalar-valued, and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are some basis vectors of $\mathbb{R}^3$.

Let $r : \mathbf{I} \to \mathbb{R}^n$, where $\mathbf{I}$ is an interval over $\mathbb{R}$. Then, the set $C = \{\mathbf{r}(t) | t \in \mathbf{I}\}$ is a *curve*. $\mathbf{r}(t)$ can be viewed as the position of a particle at time t, tracing out the curve. $\mathbf{r}'(t)$ then represents velocity, $\mathbf{r}''(t)$ represents acceleration, and so on.

The parametrisation of a curve is not unique. Additionally, if asked to parametrise a curve, ensure your parametrisation traces out the curve in the correct direction if one is specified. If a direction is specified, the curve is *oriented*, and *non-oriented* otherwise.

Example. Parametrise the semi-circle $x^2 + y^2 = 4, y \geq 0$ in the anticlockwise direction.

$$\begin{aligned} x^2 + y^2 &= 4 \\ y^2 &= 4 - x^2 \\ y &= \sqrt{4 - x^2} \end{aligned}$$

Now, let $x = t$ so we have,

$$\mathbf{r}(t) = (t, \sqrt{4 - t^2})$$

We also need bounds for t. $x^2 + y^2 = 4$, so x should range from -2 to 2, giving $t \in [-2,2]$. But now we need to verify that the curve is being traced out in the right direction. $\mathbf{r}(-2) = (-2,0)$, so we appear to be starting at the wrong side. We can fix this with the following equation:

$$\mathbf{r}(t) = (-t, \sqrt{4 - t^2}), t \in [-2,2]$$

More generally, to reverse the direction of a parametrisation, $\mathbf{r}(t) = (x(t), y(t)), t \in [a, b]$, let $\mathbf{s}(t) = (x(a + b - t), y(a + b - t)), t \in [a, b]$. In this case, $a = -b$, so we just replaced t with $-t$.

$$\mathbf{r}(t) = (2\cos(t), 2\sin(t))$$

is another valid parametrisation.

If $\mathbf{r}(t)$ is infinitely differentiable, $\mathbf{r}(t)$ parametrises a *smooth* curve.

A curve parametrised by $\mathbf{r}(t), t \in [a,b]$ is *closed* if $\mathbf{r}(a) = \mathbf{r}(b)$.

A curve that does not intersect itself is *embedded* or *simple*. A curve being embedded is equivalent to its parametrisation being injective (except possibly at the endpoints, if the curve is closed).

2 Vector Calculus

To differentiate a vector-valued function, differentiate each scalar-valued component.

If $\mathbf{r}(t)$ is a parametrisation of a curve, $\mathbf{r}'(c)$ is a tangent vector to the curve at the point $t = c$.

If $\mathbf{r}'(t) \neq 0$ for all t, then the curve parametrised by $\mathbf{r}(t)$ is *regular*.

Vector differentiation is *linear* (see MA106 §2.1 for a general overview of linearity and §6 for the linearity of specifically differentiation).

If $f(t)$ is a scalar-valued function and $\mathbf{u}(t)$, $\mathbf{v}(t)$ are vector-valued functions, then,

- $f(t)\mathbf{u}(t) = f(t)\mathbf{u}'(t) + f'(t)\mathbf{u}(t)$;
- $\mathbf{u}(t) \cdot \mathbf{v}(t) = \mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t)$;
- $\mathbf{u}(t) \times \mathbf{v}(t) = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)$;
- $\mathbf{u}(f(t)) = f'(t)\mathbf{u}'(f(t))$.

Do not write $\mathbf{u}(t)\mathbf{v}(t)$ as it is unclear as to which product (cross or dot) is being applied.

Let $\mathbf{r}(t)$ be a vector-valued function such that $||r(t)|| = C$, where C is a constant. Then $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal.

Proof.

$$\begin{aligned} \mathbf{r} \cdot \mathbf{r} &= ||\mathbf{r}||^2 \\ \frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) &= \frac{d}{dt}\mathbf{r}^2 \\ \mathbf{r} \cdot \mathbf{r}' + \mathbf{r}' \cdot \mathbf{r} &= 0 \\ \mathbf{r}' \cdot \mathbf{r} &= 0 \end{aligned}$$

■

If a curve is parametrised by $\mathbf{r}(t)$, the length of the curve between two given values of t, a and b, is given by,

$$\int_a^b ||\mathbf{r}'||\, dt \tag{1}$$

with $b \geq a$. Note that this returns a scalar, but in contrast:

The *arc length function* is given by the same integral, but with a new variable in the top bounds:

$$\mathbf{s}(t) = \int_a^t ||\mathbf{r}'(u)||\, du \tag{2}$$

This is a function, dependent on t.

The *arc length parametrisation* of $\mathbf{r}(t)$ is denoted $\mathbf{r}(s)$, where s is the arc length function. If $\mathbf{r}(t)$ is regular, then $||\mathbf{r}'(s)|| = 1$. This function may also be called the *unit-speed parametrisation.*

2.1 Curvature & Torsion

To measure curvature, we define a quantity as follows:

$$\kappa(s) = ||\mathbf{r}''(s)|| \tag{3}$$

where $\mathbf{r}(s)$ is a unit speed parametrisation.

The greater the value of $\kappa(s)$, the more curvature the curve has.

Example. Let $\mathbf{r}(t) = \mathbf{a}t + \mathbf{b}$, where $||\mathbf{a}|| = 1$. Find the curvature.

$\mathbf{r}(t)$ clearly parametrises a line, so we should expect our measure of curvature to be 0.

$\mathbf{r}'(t) = \mathbf{a}, ||\mathbf{r}'(t)|| = ||\mathbf{a}|| = 1$, so $\mathbf{r}(t)$ is unit speed, so we may use our curvature equation.

$\kappa(s) = ||\mathbf{r}''(t)|| = 0$, so a line has zero curvature as expected.

The curvature of a circle with radius $R > 0$ has constant curvature $\frac{1}{R}$ at all points. The quantity $\frac{1}{\kappa(a)}$ is called the *radius of curvature*, and represents the radius of the circle that best approximates $\mathbf{r}(s)$ at $s = a$. Such a circle is called the *osculating circle* of the curve at that point.

Curvature is independent of the choice of parametrisation: it is an intrinsic property of a curve.

For a non-unit-speed regular parametrisation, we alternatively have:

$$\begin{aligned}\kappa(t) &= \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3} \\ &= \frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\end{aligned}$$

where $\mathbf{T}(t)$ is the unit tangent.

2.2 Principle Normal & Binormal Vectors

As discussed earlier, the tangent vector of a curve is given by the derivative of its parametrisation. The *unit tangent* , $\mathbf{T}(t)$, is the normalised tangent vector.

We define the principle normal vector, $\mathbf{N}(s)$ as the vector that satisfies,

$$\mathbf{r}''(s) = \kappa(s)\mathbf{N} \text{ or equivalently, } \mathbf{T}'(s) = \kappa(s)\mathbf{N}(s), \text{ or } \kappa(s) = \mathbf{T}'(s) \cdot \mathbf{N}(s)$$

If $\kappa(s) = 0$, then the normal is undefined. The normal is perpendicular to the tangent, and points towards the centre of the osculating circle of the curve.

The *binormal* is defined as:

$$\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$$

$\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are always orthogonal and form an orthonormal basis of $\mathbb{R}^3$. The basis $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is called the *Frenet-Serret frame.*

We define *torsion*, τ as,

$$\mathbf{B}'(s) = -\tau\mathbf{N}(s)$$

or,

$$\tau = -\mathbf{B}'(s) \cdot \mathbf{N}(s)$$

Torsion is also independent of the choice of parametrisation.

Most of the results from the previous two sections can be compactly summarised as a matrix equation as follows:

$$\begin{aligned}\begin{bmatrix}\mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}'\end{bmatrix} &= \begin{bmatrix}0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{T} \\ \mathbf{N} \\ \mathbf{B}\end{bmatrix} \\ &= \begin{bmatrix}\kappa\mathbf{N} \\ -\kappa\mathbf{T} + \tau\mathbf{B} \\ -\tau\mathbf{N}\end{bmatrix}\end{aligned}$$

where $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are understood to be functions of the unit-speed function of t.

3 Multivariable Scalar-Valued Functions

A *multivariable scalar-valued function* maps vectors to scalars. For example, $f(x,y) = \sqrt{x^2 + y^2}$ is a multivariable scalar-valued function.

For a general function, $f : \mathbb{R}^3 \to \mathbb{R}$, a set of points $f(x,y,\cdots) = C$ where C is a constant is called a *level set*. When $n = 2$, level sets are also called *contour lines*. When $n = 3$, level sets are also called *isosurfaces*. For the function above, we may also represent f as a *surface* in $\mathbb{R}^3$ by setting $z = f(x,y)$.

Multivariable functions require multiple derivatives to fully describe rates of change in every direction - one for each dimension. For this, we use partial derivatives.

The partial derivative of a function $f(x,y,z)$ with respect to x, is variously written as,

$$\frac{\partial f}{\partial x}, f_x, \partial_x f$$

Other notations exist, but these are the main ones used in MA133 and MA134.

The second-order partial derivative of f with respect to x is written as,

$$\frac{\partial^2 f}{\partial x^2}, f_{xx}, \partial_{xx} f, \partial_x^2 f$$

and the second-order mixed derivative of f with respect to x, then y is given by,

$$\frac{\partial^2 f}{\partial y \partial x}, f_{xy}, \partial_{yx} f, \partial_y \partial_x f$$

Let f, g be functions of $(x,y,\cdots)$. Then,

$$\frac{\partial}{\partial x}(fg) = f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}$$

Let f be a function of (x,y), and x, y be functions of t. Then,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Let $f : \mathbb{R}^n \to \mathbb{R}$. We define the *gradient* of f, denoted grad f or ∇f, as the vector, $(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \cdots, \frac{\partial f}{\partial x_n})$. ∇ is the *grad operator*, and is effectively a vector full of differential operators. ∇ takes a scalar, $f(x,y,z)$, and returns a vector, ∇f.

The *directional derivative* of f in the direction of a unit vector, $\mathbf{v}$ is given by $D = \mathbf{v} \cdot \nabla f$. The directional derivative is the rate of change of f in the direction of $\mathbf{v}$.

3.1 Linear Approximations

Recall from A-level that the linear Taylor approximation of a function $f(x)$ about a point $x = a$ is given by $f(a) + f'(a)(x - a)$. We can similarly approximate a surface in $\mathbb{R}^3$ as a plane using partial derivatives.

A function $f(x,y)$ at the point (a,b) has a *linear approximation* given by

$$f(a,b) + \left[\frac{\partial f}{\partial x}(a,b)\right](x - a) + \left[\frac{\partial f}{\partial y}(a,b)\right](y - b)$$

This generalises to scalar valued functions of any number of variables. In general, the linear approximation of a function, $f : \mathbb{R}^n \to \mathbb{R}$ about a point $\mathbf{a}$, is given by,

$$f(\mathbf{x}) = f(\mathbf{a}) + [\nabla f(\mathbf{a})](\mathbf{x} - \mathbf{a})$$

where $\nabla f(\mathbf{a})$ is ∇f evaluated at $\mathbf{a}$.

Example. Find the linear approximation of $f(x,y) = \sqrt{1 - x^2 - y^2}$ about the point (0,0).

Setting $z = f(x,y)$, we have $z = \sqrt{1 - x^2 - y^2}$, so $x^2 + y^2 + z^2 = 1$, which is the upper half of the sphere of radius 1 centred on the origin, so we should expect our linear approximation to be a horizontal plane parallel to the $x - y$ plane.

$$\begin{aligned} \nabla f &= \left(-\frac{x}{\sqrt{1-x^2-y^2}}, -\frac{y}{\sqrt{1-x^2-y^2}} \right) \\ \nabla f(0,0) &= 0 \\ f(0,0) &= 1 \end{aligned}$$

so $f(x,y)$ is approximately 1 near (0,0). This corresponds to the plane $z = 1$ approximating the hemisphere of radius 1.

You may be asked to find the normal vector to a curve at a given point. To do this, use the fact that $\nabla f(a,b)$ is normal to $f(a,b)$.

Example. Let $f(x,y) = x + 2\sin(x + y)$. Find the normal to the curve $f(x,y) = 0$ at the point (0,0).

$$\begin{aligned} \nabla f &= (1 + 2\cos(x + y), 2\cos(x + y)) \\ \nabla f(0,0) &= (3,2) \end{aligned}$$

so the normal is (3,2), or any vector along the line $y = \frac{2x}{3}$.

3.2 Critical Points

A *critical point* of a function, $f(x)$ is a point where $\frac{df}{dx} = 0$. Critical points can be classified using the *second derivative test.*

Suppose $f'(x) = 0$ at $x = a$. If,

- $f''(a) > 0$, the critical point is a local minimum;
- $f''(a) < 0$, the critical point is a local maximum;
- $f''(a) = 0$, the test is inconclusive.

Note: Say $x = a$ is a local minimum/maximum point or that $f(a)$ is a local maximum.

Now, we can extend this definition to functions of two variables. A function, $f(x,y)$ has a critical point at (a,b) if $\nabla f(a,b) = 0$. That is to say, *every* partial derivative has to evaluate to 0 at the given point. We can similarly classify these critical points using the *Hessian matrix* and the *second partial derivative test.*

$$\mathbf{H} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

Let $D = \det(\mathbf{H}) = f_{xx}f_{yy} - f_{xy}^2$ ($f_{xy} = f_{yx}$ by Young's theorem) Suppose $\nabla f(x,y) = 0$ at (a,b). If,

- $D > 0$ and $f_{xx} > 0$ at (a,b), then (a,b) is a local minimum point;
- $D > 0$ and $f_{xx} < 0$ at (a,b), then (a,b) is a local maximum point;
- $D < 0$ at (a,b), then (a,b) is a saddle point;
- $D = 0$ at (a,b), then the test is inconclusive.

If it is easier to calculate, you may check f_{yy} instead of f_{xx} for the first two cases.

Performing this test on a single-variable function just gives the standard second-derivative test.

For functions or three or more variables, the determinant alone does not provide sufficient information to classify the critical point. Instead, we check the eigenvalues of the Hessian.

Suppose $f : \mathbb{R}^{\ltimes} \to \mathbb{R}, n > 3$ and $\nabla f(\mathbf{x}) = 0$ at $\mathbf{x} = \mathbf{a}$. If the Hessian has,

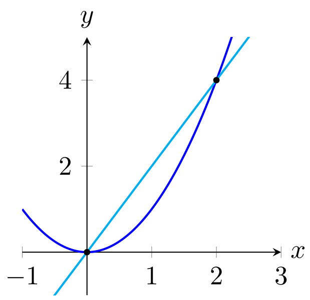
- all eigenvalues positive at $\mathbf{a}$, then $\mathbf{a}$ is a local minimum point;
- all eigenvalues negative at $\mathbf{a}$, then $\mathbf{a}$ is a local maximum point;
- both positive and negative eigenvalues at $\mathbf{a}$, then $\mathbf{a}$ is a saddle point;
- other; the test is inconclusive.

4 Integration

Most of these methods are best explained through example.

4.1 Double Integration

Example. Write down the area between $y = x^2$ and $y = 2x$ as a double integral in two different orders.



We observe that the points of intersection are (0,0) and (2,4), which will be helpful for some of our integration bounds.

Let's do the order $dy\,dx$ first. Working from the outermost integral to the innermost integral: We're looking at dx first - the change in x. x varies between 0 and 2, so our outermost integral should go from 0, up to 2. Now, how does y vary? On the graph, x^2 is below $2x$, so we have x^2 as our lower bound and $2x$ as our upper. Overall, we have

$$\int_0^2 \int_{x^2}^{2x} 1\,dy\,dx$$

Doing the other order is very similar. y varies from 0 to 4, so our outermost integral goes from 0 to 4. Now, how does x vary? Well, the line $y = 2x$ is "below" the line $y = x^2$ (taking right to be the positive "upwards" direction), so we should go from $y = 2x$ to $y = x^2$. But we're integrating with respect to x, so we can't have x in our integration bounds, so rearrange for y to get $x = \frac{y}{2}$ and $x = \sqrt{y}$ as our bounds. Overall, we have

$$\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} 1\,dx\,dy$$

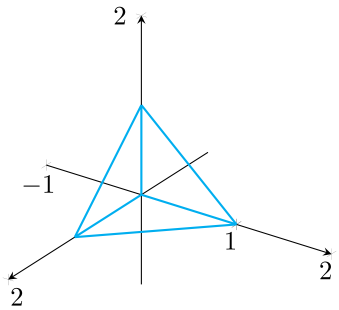
Note that, for a double integral for working out area, your outermost bounds should always be numbers and not functions of any of the variables being integrated with respect to. If the sole integrand is 1, it also may be tastefully omitted.

More generally, the double integral of some function $\iint_R f(x,y)\,dA$ evaluated over some region, R, represents the volume between $f(x,y)$ and the $x-y$ plane over the region. If $f(x,y) = 1$ as above, then this is just finding the area of the region.

You can also think of these multiple integrals in terms of finding masses, with $f(x,y)$ being some kind of density function. $f(x,y) = 1$ would then represent a constant density, just giving area in the 2D case.

4.2 Triple Integration

Example. Write down the volume contained within the tetrahedron with vertices (0,0,0), (1,0,0), (0,1,0) and (0,0,1) as a triple integral.



Let's integrate with the equilateral face being a function of x and y over the triangular region in the $x-y$ plane bounded by (0,0,0), (1,0,0) and (0,1,0). (Looking down at the tetrahedron from above, everything is contained within the triangle bounded by those three points.)

The lines of interest are the x and y axes, and $x+y=1$.

Using our previous method, x varies from 0 to 1 in this region, so our outermost integral goes from 0 to 1. Now, how does y vary in terms of x? The line $x+y=1$ is above the x axis here, so we go from $y=0$ to $x+y=1$, but again, no x's in our bounds, so rearrange for $y=1-x$, and we have our two outermost integrals sorted.

Now, just look at the z-axis. How does z vary in terms of x and y? The $x-y$ plane is below the plane $x+y+z=1$, so we go from the $x-y$ plane, which is given by $z=0$, to the plane $x+y+z=1$, which is rearranged to $z=1-x-y$. Overall, we have,

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dx$$

4.3 Change of Coordinate System

4.3.1 Polar Coordinates

In $\mathbb{R}^2$, the *Cartesian coordinates*, (x,y), and *polar coordinates*, (r,θ) are related by,

$$x = r\cos\theta$$
$$y = r\sin\theta$$

We may convert the double integral,

$$\iint_R f(x,y)\,dA = \int\int f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

Remember to multiply by an extra r.

In general, if you see many (x^2+y^2)'s in $f(x,y)$, converting to polar coordinates might be a good idea.

Example. Find the area in the first quadrant bounded by the polar curve $r = 1 + \cos\theta$.

For polar double integrals, it is typically easier to use the angle in the outermost integral. So, using our formula from before, $f(x,y) = 1$, so we have,

$$\int\int r\,dr\,d\theta$$

But what are our bounds? We're looking for bounds in terms of angles, so clearly the first quadrant is bounded by $\theta = 0$ and $\theta = \frac{\pi}{2}$. Now, how does r vary? In this case, it's fairly simple, as r just goes from 0 up to the given curve, so we have,

$$\int_0^{\frac{\pi}{2}}\int_0^{1-\cos\theta} r\,dr\,d\theta$$

4.3.2 Cylindrical Coordinates

In $\mathbb{R}^3$, the Cartesian coordinates, (x,y,z), and *cylindrical coordinates*, (r,θ,z) are related by,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

We may convert the triple integral,

$$\iiint_\Omega f(x,y,z)\,dV = \int\int\int f(r\cos\theta,r\sin\theta,z)r\,dr\,d\theta\,dz$$

<u>Remember to multiply by an extra r.</u>

Again, if you see many $(x^2 + y^2)$'s in $f(x,y,z)$, converting to cylindrical coordinates might be a good idea.

Example. Find the value of

$$\iiint_\Omega (x^2 + y^2)\,dV$$

where Ω is the region bounded by the surfaces $x^2 + y^2 = 1$, $z = 2 - x$, and $z = 0$.

First, plug in our formula:

$$\iiint_\Omega x^2 + y^2\,dV = \iiint_\Omega (r\cos\theta)^2 + (r\sin\theta)^2 r\,dr\,d\theta\,dz$$

Simplify the sines and cosines to get,

$$\iiint_\Omega r^3\,dr\,d\theta\,dz$$

which certainly looks easier to do. But what are our bounds of integration? As we said earlier, it's easier to work with the angles on the outside, so let's swap the order of integration to $dr\,dz\,d\theta$. The surfaces given are now $r = 1$, $z = 2 - r\cos\theta$ and $z = 0$. $r = 1$ takes all angles, so we go from 0 to 2π for our outermost integral. Next, how does r vary? Well, r just goes from 0 (the z-axis) out to the curve $r = 1$, so we go from 0 to 1 for the middle integral. Finally, we look at how z varies. We are given two surfaces for z: $z = 2 - r\cos\theta$ and $z = 0$. As r is at most 1 over this region, $2 - r\cos\theta > 0$, so we know that z goes from 0 to $2 - r\cos\theta$. Overall, we have,

$$\int_0^{2\pi}\int_0^1\int_0^{2-r\cos\theta} r^3\,dr\,dz\,d\theta$$

Skipping over the actual integration, you find that this evaluates to π.

4.3.3 Spherical Coordinates

In $\mathbb{R}^3$, the Cartesian coordinates, (x,y,z), and *spherical coordinates*, (r,θ,ϕ) are related by,

$$\begin{aligned} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi \end{aligned}$$

(You can think of it like applying a cylindrical coordinate transformation to x, y and z, then polar coordinates to (x,y) and z)

Like in polar/cylindrical coordinates, θ is measured from the positive x-axis. ϕ is measured from the positive z-axis. Notice that, unlike θ, ϕ only ever varies from 0 to π. If ϕ goes past that, then the same point can be reached with a smaller angle, just by increasing θ by π radians.

We may convert the triple integral,

$$\iiint_{\Omega} f(x,y,z)\,dV = \int\int\int f(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)r^2\sin\phi\,dr\,d\theta\,dz$$

Remember the extra factor of $r^2\sin\phi$.

Similarly, if you see many $(x^2+y^2+z^2)$'s in $f(x,y,z)$, converting to spherical coordinates might be a good idea.

Example. Find a triple integral expression for the volume of a sphere of radius R centred on the origin.

Since we're finding a volume, $f(x,y,z)=1$, so $f(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)=1$. As before, it's easier to work with when angles are in the outermost integrals, let's do $dr\,d\theta\,d\phi$ for our order of integration. First, how does ϕ vary? Looking at the sphere, notice that it contains both the positive and negative z-axis, so ϕ varies from 0 to π (the sphere contains points directly above and below the origin). next, look straight down at the $x-y$ plane from the positive z-axis. The sphere lies in all 4 quadrants, so θ takes all values possible, varying from 0 to 2π. Finally, r goes from 0 up to the surface of the sphere which lies at a constant distance R away from the origin. Overall, we have,

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{R} dr\,d\theta\,d\phi$$

Example. Find the value of

$$\iiint_{\Omega} z^2\,dV$$

where Ω is the region bounded by two spheres of radius 1 and 2 centred at the origin.

Given that we have two spheres, doing this in Cartesian coordinates is a horrible idea. So we convert our dV into $r^2\sin\phi\,dr\,d\theta\,d\phi$ and use spherical coordinates.

$$\iiint_{\Omega} dV = \iiint_{\Omega} (r\cos\phi)^2 r^2\sin\phi\,dr\,d\theta\,d\phi$$

We just worked out the bounds for a spherical region, so we can mostly just plug them in. However, we're looking for the volume between two spheres, so we look at how r varies in this situation. Thinking about the graph from the perspective of r, the sphere of radius 1 is "below" (closer to the origin) than the sphere of radius 2, so r varies from 1 to 2. So, overall, we have,

$$\int_0^{\pi}\int_0^{2\pi}\int_1^{2} r^4\cos^2\phi\sin\phi\,dr\,d\theta\,d\phi$$

Example. Find the value of

$$\iiint_\Omega \sqrt{\exp(x^2+y^2+z^2)^3}\,dV$$

where Ω is the region bounded by $x^2+y^2+z^2=1$.

We use the same conversion again, noting that the volume being integrated over is a sphere. Also note that $x^2+y^2+z^2=r^2$, so we can write,

$$\iiint_\Omega \sqrt{\exp(x^2+y^2+z^2)^3}\,dV = \int_0^\pi \int_0^{2\pi} \int_0^1 sqrt\exp(r^2)^3\,dr\,d\theta\,d\phi$$

4.3.4 Arbitrary Change of Coordinates

The *Jacobian matrix* of a function, $\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^n$, denoted $D\mathbf{f}$, is the matrix of partial derivatives,

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \cdots & \frac{\partial f_3}{\partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \frac{\partial f_n}{\partial x_3} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

This can be more compactly written as,

$$\begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix}$$

or,

$$\begin{bmatrix} \nabla^{\mathrm{T}} f_1 \\ \vdots \\ \nabla^{\mathrm{T}} f_n \end{bmatrix}$$

If $\mathbf{F}:\mathbb{R}^2\to\mathbb{R}^2$ is a bijection defined by $\mathbf{F}(u,v)=(x=a(u,v),y=b(u,v))$ for some functions, a and b, then the integral of $f:A\subseteq\mathbb{R}^2\to\mathbb{R}$ can be given by,

$$\iint_A f(x,y)\,dx\,dy = \iint_B f(u,v)|\det(\mathbf{J}(\mathbf{F}(u,v)))|\,du\,dv$$

where $\mathbf{J}$ is the Jacobian of the function $\mathbf{F}(u,v)$, and B is the same region in the new coordinate system.

From this point onwards, when the function is obvious and there is little room for confusion, I will write $\mathbf{J}$ to denote $\mathbf{J}(\mathbf{F}(\mathbf{x}))$.

Omitting the function $f(x,y)$, we can more compactly write $dA=dx\,dy=|\det(\mathbf{J})|\,du\,dv$.

For a triple integral, we similarly have, $dV=dx\,dy\,dz=|\det(\mathbf{J})|\,du\,dv\,dw$.

Example. Verify the area and volume element conversions previously found using the formula outlined above.

For Cartesian $\mathbb{R}^2$ to polar, we have $\mathbf{F}(r,\theta) = (x = r\cos\theta, y = r\sin\theta)$.

$$\begin{aligned}\mathbf{J} &= \begin{bmatrix}\nabla x\\ \nabla y\end{bmatrix}\\ &= \begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix}\\ &= \begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}\end{aligned}$$

$\det(\mathbf{J}) = r(\cos\theta)^2 + r(\sin\theta)^2 = r((\cos\theta)^2 + (\sin\theta)^2) = r$, so $dA = r\,dr\,d\theta$, as we found before.

For Cartesian $\mathbb{R}^3$ to cylindrical, we have $\mathbf{F}(r,\theta,z) = (x = r\cos\theta, y = r\sin\theta, z = z)$.

$$\begin{aligned}\mathbf{J} &= \begin{bmatrix}\nabla x\\ \nabla y\\ \nabla z\end{bmatrix}\\ &= \begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z}\end{bmatrix}\\ &= \begin{bmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\end{aligned}$$

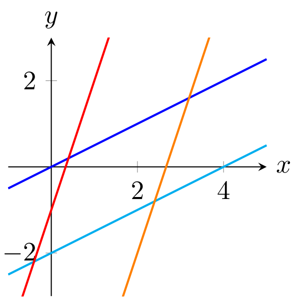
By Sarrus' rule, $\det(\mathbf{J}) = r(\cos\theta)^2 + r(\sin\theta)^2 = r((\cos\theta)^2 + (\sin\theta)^2) = r$, so $dV = r\,dr\,d\theta\,dz$ as before.

Spherical coordinates are similar, but algebraically involved and time consuming, so they will be omitted.

Example. Evaluate

$$\iint_R \frac{x-2y}{3x-y}\,dA$$

where R is the region enclosed by the lines $y = \frac{x}{2}$, $y = \frac{x}{2} - 2$, $y = 3x - 1$ and $y = 3x - 8$.



Let $u = x - 2y$ and $v = 3x - y$. Rearranging for x and y, we have $x = \frac{2v}{5} - \frac{u}{5}$ and $y = \frac{v}{5} - \frac{3u}{5}$.

Let $\mathbf{F}(x,y) = (x = \frac{2v}{5} - \frac{u}{5}, y = \frac{v}{5} - \frac{3u}{5})$.

$$\mathbf{J} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{bmatrix}$$

$\det(\mathbf{J}) = \frac{1}{5}$, so we have,

$$\iint\limits_R \frac{x-2y}{3x-y}\, dA = \iint\limits_S \frac{u}{v}\frac{1}{5}\, du\, dv$$

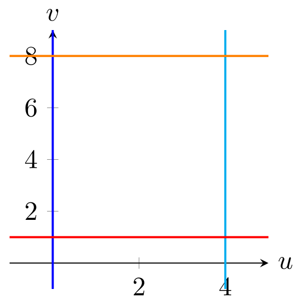
Now to find the bounds. Rearrange the lines defining the region:

$$y = \frac{x}{2} \Leftrightarrow x - 2y = 0 \Leftrightarrow u = 0$$

$$y = \frac{x}{2} - 2 \Leftrightarrow x - 2y = 4 \Leftrightarrow u = 4$$

$$y = 3x - 1 \Leftrightarrow 3x - y = 1 \Leftrightarrow v = 1$$

$$y = 3x - 8 \Leftrightarrow 3x - y = 8 \Leftrightarrow v = 8$$



So we have $u = 0$, $u = 4$, $v = 1$, and $v = 8$ as our bounds.

$$\int_1^8 \int_0^4 \frac{u}{5v}\, du\, dv$$

5 Vector Fields

Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar-valued function, assigning a scalar to every point in $\mathbb{R}^n$. Such a function may also be called a *scalar field*. Also recall that a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is vetor-valued, assigning a m-vector to every point in $\mathbb{R}^n$. Such a function may be called a *vector field*. We now consider the case for $n = m = 2$ and $n = m = 3$.

If a vector field, $\mathbf{F}(x,y)$ can be written as the gradient of some scalar valued function, $f(x,y)$, then $\mathbf{F}$ is a vector field whose flow is normal to the level sets of f. Such a field is called a *conservative* field.

For example, the vector field $\mathbf{F}(x,y) = (x,0)$ can also be written as the gradient of the function $f(x,y) = \frac{x^2}{2}$, so $\mathbf{F}$ is conservative.

5.1 Divergence & Curl

In this section, we will mostly be considering functions from $\mathbb{R}^3$ to $\mathbb{R}^3$. Recall the del operator, ∇, which we previously used to write the gradient function. We now use the same symbol to introduce some new operators:

$$\begin{aligned} \operatorname{grad} f &= \nabla f \\ \operatorname{div} \mathbf{F} &= \nabla \cdot \mathbf{F} \\ \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} \end{aligned}$$

These last two operators look rather strange, given that del is an operator and not a vector, but the way we've written them is a useful mnemonic as to how we actually calculate them: pretend the del is a vector full of partial differential operators, and perform the calculation as indicated. Whenever you multiply a function by a differential operator, instead apply the operator to the function.

Let $\mathbf{F}(x,y,z) = (f(x),g(y),h(z))$. Then,

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \text{``}\cdot\text{''} \begin{bmatrix} f \\ g \\ h \end{bmatrix} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \text{``}\times\text{''} \begin{bmatrix} f \\ g \\ h \end{bmatrix} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{bmatrix} = \begin{bmatrix} \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial z} - \frac{\partial g}{\partial x} \\ \frac{\partial h}{\partial x} - \frac{\partial g}{\partial y} \end{bmatrix}$$

The grad operator takes a scalar and returns a vector. The div operator takes a vector and returns a scalar. The curl operator takes a vector and returns a vector.

The curl of a conservative field is 0.

5.2 Parametric Surfaces

Recall that a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be parametrised with a one variable function. A surface in $\mathbb{R}^3$ can similarly be parametrised using a two variable function. You will have seen some of these previously at A-level, such as in the double vector parametrisation of a plane.

Example. Parametrise the surface in $\mathbb{R}^3$ given by $x^2 + y^2 + z^2 = A^2$.

This is clearly a sphere, so try spherical coordinates with $r = A$. Then, $x = A\cos u \sin v$, $y = A \sin u \sin v$, $z = A\cos v$, with $u \in [0,2\pi), v \in [0,\pi]$. Remember to give intervals for your parameters.

Example. Find the Cartesian equation of the surface parametrised by $\mathbf{r}(u,v) = (\sqrt{1-u}\cos v, \sqrt{1-u}\sin v, u)$, $u \in (-\infty, 1], v \in [0, 2\pi]$.

$x = \sqrt{1-u}\cos v$, $y = \sqrt{1-u}\sin v$ and $z = u$. Now look for ways to eliminate things. For example, $x^2 + y^2 = (1-u)\cos^2 v + (1-u)\sin^2 v = 1 - u = 1 - z$, so we have $x^2 + y^2 + z = 1$ for our Cartesian equation.

5.3 Surface Integrals

$$dS = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\, dv$$

Note: you only need the length of the cross product and not the vector itself, so to save time, you can check if $\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} = 0$. If so, then you can instead use $\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| = \left|\frac{\partial \mathbf{r}}{\partial u}\right|\left|\frac{\partial \mathbf{r}}{\partial v}\right|$.

Example. Find the formula for the surface area element for an arbitrary function $z = f(x,y)$.

$$\begin{aligned}
\mathbf{r}(x,y) &= \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix} \\
\frac{\partial \mathbf{r}}{\partial x} &= \begin{bmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x} \end{bmatrix} \\
\frac{\partial \mathbf{r}}{\partial y} &= \begin{bmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y} \end{bmatrix} \\
\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} &= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial f}{\partial y} \\ 1 \end{bmatrix}
\end{aligned}$$

so $\left|\frac{\partial f}{\partial y} \times \frac{\partial f}{\partial v}\right| = \sqrt{\frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2 + 1}$, and,

$$\iint dS = \int\int \sqrt{\frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2 + 1}\, dx\, dy$$

5.4 Divergence Theorem

Flux is a vector quantity that describes how a fluid would flow tbrough a surface, S, of a volume, V. You don't really need to know what it is, since the only questions you'll be asked about it will basically just say "calculate the flux", and then boil down to doing an integral.

The *divergence theorem* says,

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

where $\mathbf{n}$ is the outward-pointing unit normal to the surface S. Note, instead of S, sometimes, ∂V is written for the surface area element. Also, for the flux integral, sometimes $\mathbf{F} \cdot d\mathbf{S}$ is written instead of $\mathbf{F} \cdot \mathbf{n}\, dS$. $d\mathbf{S}$ is also equal to $\pm(\mathbf{r}_u \times \mathbf{r}_v)\, dy\, dv$. This form may be easier to caclulate, as you avoid finding the normal vector.

Example. Let V be the volume bounded by the unit sphere, and let $\mathbf{F}(x,y,z) = (z,y,x)$. Calculate the net flux over the surface of the sphere.

First, write the equation for flux

$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint\limits_V \nabla\cdot\mathbf{F}\,dV$$
$$= \iiint\limits_V \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} z \\ y \\ x \end{bmatrix} dV$$
$$= \iiint\limits_V 0+1+0\,dV$$
$$= \iiint\limits_V dV$$

which is just the volume of the unit sphere, which is $\frac{4\pi}{3}$.

Without the divergence theorem, the original integral is still doable with spherical coordinates, but is a lot more work to compute.

5.5 Line Integrals

Given a function, $\mathbf{F}:\mathbb{R}^n\to\mathbb{R}^n$, and a curve, $C=\mathbf{r}(t)$,$t\in[a{,}b]$, the integral of $\mathbf{F}\cdot d\mathbf{r}$ over C is called a *line integral.* In practice, we can calculate it as follows:

$$\oint\limits_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt$$

The line integral of a curve across a force vector field represents the work done when moving an object along the curve C from where $t=a$ to $t=b$.

Example. Calculate the line integral of $\mathbf{F}(x{,}y)=(x^2,-xy)$ along the unit circle from (1,0) to (0,1) in the anticlockwise direction.

First, parametrise the path being integrated along: $\mathbf{r}(t)=(\cos t,\sin t)$, $t\in[0,\frac{\pi}{2}]$. $\mathbf{r}'(t)=(-\sin t,\cos t)$. Now, write out the line integral, and replace the x and y with $\cos(t)$ and $\sin(t)$ in $\mathbf{F}$.

$$\oint\limits_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\frac{\pi}{2}} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt$$
$$= \int_0^{\frac{\pi}{2}} \begin{bmatrix} \cos^2 t \\ -\cos t\sin t \end{bmatrix}\cdot\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} dt$$
$$= \int_0^{\frac{\pi}{2}} -\cos^2 t\sin t - \cos^2 t\sin t\,dt$$
$$= -2\int_0^{\frac{\pi}{2}} \cos^2 t\sin t\,dt$$
$$= -\frac{2}{3}$$

The line integral between two given points of a conservative vector field is independent of the path taken.

For example, gravity is a conservative field; it doesn't matter how you climb up a mountain, you gain the same amount of gravitational potential energy regardless of choice of path. This also means that the line integral of a closed curve is zero. Going back to the gravitational example, if you end up back where you started, you'll have a net gain of 0 gravitational potential energy.

5.6 Circulation

We haven't talked much about curl yet, but its name gives a hint as to what physical characteristic of a field it may represent. Thinking of $\mathbf{F}(x,y,z)$ as the function that returns the velocity of a fluid, we can quantify the pointwise rotation of the fluid at any given point with the curl of the vector field evaluated at that point. The length of the curl is proportional to the speed of rotation, and its direction is normal to the plane of rotation.

If the point at which curl is being evaluated lies on a surface, S, with unit normal $\mathbf{n}$, we can define a quantity called pointwise circulation as, $\nabla \times \mathbf{F} \cdot \mathbf{n}$. Unlike curl, this is a scalar, whose sign indicates the direction of rotation around the point, relative to $\mathbf{n}$.

Similar to the corkscrew rule for the cross product, if you align your right hand thumb in the direction of $\mathbf{n}$, the direction of circulation flows along with your fingers if pointwise circulation is positive, and flows against your fingers if pointwise circulation is negative.

By integrating the circulation over the entire surface, we can find the net circulation over the surface.

5.6.1 Stokes' Theorem

Let $\mathbf{F}(x,y,z)$ be a vector field, S be a surface with unit normal $\mathbf{n}$ and boundary curve C, oriented according to the right-hand rule. Then, the net circulation of the surface S over the field $\mathbf{F}$ is equal to the line integral of $\mathbf{F}$ evaluated over C.

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

Example. Calculate the integral of $\nabla \times \mathbf{F} \cdot dS$ over the surface $S : z = 4 - x^2 - y^2, z \in [0,4]$.

The rim of the surface is a circle of radius 2 centred on the origin, which can be parametrised as, $\mathbf{r}(t) = (2\cos t, 2\sin t, 0), t \in [0,2\pi]$. $\mathbf{r}'(t) = (-2\sin t, 2\cos t)$. We then rewrite the vector field as $(2\sin t, 0, 4\cos^2 t)$.

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \, dt &= \int_C \begin{bmatrix} 2\sin t \\ 0 \\ 4\cos^2 t \end{bmatrix} \cdot \begin{bmatrix} -2\sin t \\ 2\cos t \\ 0 \end{bmatrix} dt \\
&= -4 \int_0^{2\pi} \sin^2 t \, dt \\
&= -4\pi
\end{aligned}$$

6 Closing Remarks & Condensed Summary

As was mentioned in the introduction, this module is extremely computational in nature, and doesn't require a lot of conceptual understanding. Do some practice if you haven't integrated in a while, and you'll probably be fine.

If a direction is specified along a curve, the curve is *oriented* . To reverse the direction of a parametrisation, $\mathbf{r}(t) = (x(t),y(t)),t \in [a,b]$, let $\mathbf{s}(t) = (x(a+b-t),y(a+b-t)),t \in [a,b]$ If $\mathbf{r}(t)$ is infinitely differentiable, $\mathbf{r}(t)$ parametrises a *smooth* curve. A curve that does not intersect itself is *embedded* or *simple.* A curve parametrised by $\mathbf{r}(t),t \in [a,b]$ is *closed* if $\mathbf{r}(a) = \mathbf{r}(b)$. If $\mathbf{r}'(t) \neq 0$ for all t, $\mathbf{r}(t)$ parametrises a *regular* curve.

If a curve is parametrised by $\mathbf{r}(t)$, the length of the curve between two given values of t, a and b, is given by,

$$\int_a^b ||\mathbf{r}'||\,dt \tag{4}$$

with $b \geq a$. This is a scalar.

The *arc length function* is given by the same integral, but with a new variable in the top bounds:

$$\mathbf{s}(t) = \int_a^t ||\mathbf{r}'(u)||\,du \tag{5}$$

This is a function, dependent on t.

The *arc length parametrisation* of $\mathbf{r}(t)$ is denoted $\mathbf{r}(s)$, where s is the arc length function. If $\mathbf{r}(t)$ is regular, then $||\mathbf{r}'(s)|| = 1$. This function may also be called the *unit-speed parametrisation.*

Finding the frenet-serret frame:

- $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$
- $\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$
- $\mathbf{B} = \mathbf{T} \times \mathbf{N}$
- $\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|}$
- $\tau = -\mathbf{B}' \cdot \mathbf{N}|\mathbf{r}'|$

If $\mathbf{r}$ is a regular unit-speed parametrisation, then $|\mathbf{r}'| = 1$, so just ignore wherever it appears in the formulae above.

Matrix form relating the frenet-serret vectors:

$$\begin{bmatrix}\mathbf{T}'\\ \mathbf{N}'\\ \mathbf{B}'\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix} = \begin{bmatrix}\kappa\mathbf{N}\\ -\kappa\mathbf{T}+\tau\mathbf{B}\\ -\tau\mathbf{N}\end{bmatrix}$$

$\mathbf{T}$ is the *unit tangent.*

$\mathbf{N}$ is the *normal.* It points towards the centre of the osculating circle (see below).

$\mathbf{B}$ is the *binormal.*

κ is the *curvature*.

τ is the *torsion*.

$\mathbf{T}$, $\mathbf{N}$, $\mathbf{N}$, κ and τ are all functions of t (or s, which itself is a function of t). $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are all perpendicular and form an orthonormal basis of $\mathbb{R}^3$.

$\frac{1}{\kappa}$ is the radius of curvature, representing the radius of the circle which best approximates the curve at that point. Such a circle is called an *osculating circle*.

Let $f : \mathbb{R}^n \to \mathbb{R}$. We define the *gradient* of f, denoted grad f or ∇f, as the vector, $(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \cdots, \frac{\partial f}{\partial x_n})$.

∇ is the *grad operator*, and is effectively a vector full of differential operators. ∇ takes a scalar, $f(x,y,z)$, and returns a vector, ∇f.

The *directional derivative* of f in the direction of a unit vector, $\mathbf{v}$ is given by $D = \mathbf{v} \cdot \nabla f$. The directional derivative is the rate of change of f in the direction of $\mathbf{v}$.

the linear approximation of a function, $f : \mathbb{R}^n \to \mathbb{R}$ about a point $\mathbf{a}$, is given by,

$$f(\mathbf{x}) = f(\mathbf{a}) + [\nabla f(\mathbf{a})]\,(\mathbf{x} - \mathbf{a})$$

where $\nabla f(\mathbf{a})$ is ∇f evaluated at $\mathbf{a}$.

The normal vector to this tangent plane is given by $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$.

Suppose $\nabla f(x,y) = 0$ at (a,b).

$$\mathbf{H} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

Let $D = \det(\mathbf{H}) = f_{xx}f_{yy} - f_{xy}^2$ ($f_{xy} = f_{yx}$ by Young's theorem)

If,

- $D > 0$ and $f_{xx} > 0$ at (a,b), then (a,b) is a local minimum point;
- $D > 0$ and $f_{xx} < 0$ at (a,b), then (a,b) is a local maximum point;
- $D < 0$ at (a,b), then (a,b) is a saddle point;
- $D = 0$ at (a,b), then the test is inconclusive.

If it is easier to calculate, you may check f_{yy} instead of f_{xx} for the first two cases.

Surface & Volume Elements

Cartesian:

$$dA = dx\,dy$$
$$dV = dx\,dy\,dz$$

Polar:

$$x = \cos\theta$$
$$y = \sin\theta$$
$$dA = r\,dr\,d\theta$$

Cylindrical:

$$x = \cos\theta$$

$$y = \sin\theta$$
$$z = z$$
$$dV = r\,dr\,d\theta\,dz$$

Spherical:

$$x = r\cos\theta\sin\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\phi$$
$$dV = r^2\sin\phi\,dr\,d\theta\,d\phi$$

Arbitrary change of coordinate system given by $\mathbf{F}(u,v) = (x(u,v), y(u,v))$:

$$x = x(u,v)$$
$$y = y(u,v)$$
$$dx\,dy = |det(\mathbf{J}(\mathbf{F}(u,v)))|\,du\,dv$$

Where $\mathbf{J}$ is the Jacobian matrix, given by $(\nabla x, \nabla y, \cdots)$

$$dS = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\,dv$$
$$dA = |\det(\mathbf{J})|\,du\,dv$$
$$dV = |\det(\mathbf{J})|\,du\,dv\,dw$$

$$\operatorname{grad} f = \nabla f$$
$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F}$$
$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F}$$

The grad operator takes a scalar and returns a vector. The div operator takes a vector and returns a scalar. The curl operator takes a vector and returns a vector.

If a vector field, $\mathbf{F}(x,y)$ can be written as the gradient of some scalar valued function, $f(x,y)$, then $\mathbf{F}$ is a vector field whose flow is normal to the level sets of f. Such a field is called a *conservative* field.

The curl of a conservative field is 0.

Flux is a vector quantity that describes how a fluid would flow through a surface, S, of a volume, V.

Divergence theorem: the surface integral of a vector field over a closed surface (the flux of the surface) is equal to the volume integral of the divergence over the region enclosed by the surface. That is to say,

$$\iiint_V \nabla\cdot\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS$$

where $\mathbf{n}$ is the outward-pointing unit normal to the surface S. Note, instead of S, sometimes, ∂V is written for the surface area element. Also, for the flux integral, sometimes $\mathbf{F}\cdot d\mathbf{S}$ is written instead of $\mathbf{F}\cdot\mathbf{n}\,dS$. $d\mathbf{S}$ is also equal to $\pm(\mathbf{r}_u \times \mathbf{r}_v)\,dy\,dv$. This form may be easier to caclulate, as you avoid finding the normal vector.

Given a function, $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, and a curve, $C = \mathbf{r}(t), t \in [a,b]$, the integral of $\mathbf{F} \cdot d\mathbf{r}$ over C is called a *line integral*. In practice, we can calculate it as follows:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$$

The line integral of a curve across a force vector field represents the work done when moving an object along the curve C from where $t = a$ to $t = b$.

The line integral between two given points of a conservative vector field is independent of the path taken. The line integral of a closed curve over a conservative vector field is zero.

If the point at which curl is being evaluated lies on a surface, S, with unit normal $\mathbf{n}$, we can define a quantity called pointwise circulation as, $\nabla \times \mathbf{F} \cdot \mathbf{n}$. Unlike curl, this is a scalar, whose sign indicates the direction of rotation around the point, relative to $\mathbf{n}$.

Similar to the corkscrew rule for the cross product, if you align your right hand thumb in the direction of $\mathbf{n}$, the direction of circulation flows along with your fingers if pointwise circulation is positive, and flows against your fingers if pointwise circulation is negative.

By integrating the circulation over the entire surface, we can find the net circulation over the surface.

Stokes' Theorem: Let $\mathbf{F}(x,y,z)$ be a vector field, S be a surface with unit normal $\mathbf{n}$ and boundary curve C, oriented according to the right-hand rule. Then, the net circulation of the surface S over the field $\mathbf{F}$ is equal to the line integral of $\mathbf{F}$ evaluated over C.

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

7 Additional Techniques

This section will cover further techniques for integration that you may find faster and/or easier to perform. These techniques are not examinable, but I highly recommend at least learning tabular integration by parts, as it streamlines the method taught at A-Level to an extreme degree, particularly for repeated applications of integration by parts.

7.1 Tabular Integration by Parts

Say we want to integrate this function,

$$\int a(x)b(x)\,dx$$

Being a product of two functions, we use integration by parts.

Draw out a table, with D above the first column and I above the first, then put a column of alternating plusses and minuses, besides the first, starting with a plus. You'll get a feel for how many rows you'll need as you get more used to using this method, but for now, I've drawn 4.

	D	I
$+$		
$-$		
$+$		
$-$		

Now, look at the integral, and decide which function is easier to differentiate. Or more usually, which function you don't want to integrate. Suppose we don't want to integrate $a(x)$, so we differentiate $a(x)$ and integrate $b(x)$.

Put $a(x)$ under D, and $b(x)$ under I, and differentiate and integrate them repeatedly, putting the result in the next row each time. For ease of reading, let $b_{.}(x)$ indicate the first integral of $b(x)$, $b_{..}(x)$ the second, and so on.

	D	I
$+$	$a(x)$	$b(x)$
$-$	$a'(x)$	$b_{.}(x)$
$+$	$a''(x)$	$b_{..}(x)$
$-$	$a'''(x)$	$b_{...}(x)$

When we decide to stop (I've done 3 additional rows here), multiply diagonal elements, keeping the signs attached. Then, multiply the final row horizontally and throw it into an integral;

	D		I
$+$	$a(x)$	$\times$	$b(x)$
$-$	$a'(x)$	$\times$	$b_{.}(x)$
$+$	$a''(x)$	$\times$	$b_{..}(x)$
$-$	$a'''(x)$		$b_{...}(x)$

$$\int a(x)b(x)\,dx = [+a(x)b_{.}(x)] + [-a'(x)b_{..}(x)] + [a''(x)b_{...}(x)] + \int [-a'''(x)b_{...}(x)]\,dx$$

But when do we know when to stop?

There are three main stops:

- There is a 0 in the D column.
- You can integrate a row.
- A row appears more than once.

In the first case, when you multiply the last row together, the final integral just disappears. In the second case, if you can integrate a row, just stop the process and do the integral. In the third case, if a row appears more than once, that means you can rewrite the original integral in terms of itself, plus some extra stuff at the front, which you can rearrange for.

Example. Evaluate,

$$\int x^3 \sin(4x)\,dx$$

It's almost always ideal to differentiate the polynomial, as we know we can eventually get it to 0. The sine function is fine to integrate as well, so let's do that.

	D		I
$+$	x^3	$\times$	$\sin(4x)$
$-$	$3x^2$	$\times$	$-\frac{1}{4}\cos(4x)$
$+$	$6x$	$\times$	$-\frac{1}{16}\sin(4x)$
$-$	6	$\times$	$\frac{1}{64}\cos(4x)$
$+$	0		$\frac{1}{256}\sin(4x)$

$$\int x^3 \sin(4x)\,dx = -\frac{1}{4}x^3\cos(4x) + \frac{3}{16}x^2\sin(4x) + \frac{3}{32}x\cos(4x) - \frac{3}{128}\sin(4x)$$

Example. Evaluate,

$$\int x^3 \ln x\,dx$$

We like to differentiate polynomials, but integrating $\ln x$ requires integration by parts in the first place, which we would like to avoid, especially if we are repeatedly integrating it. So, we differentiate $\ln x$ and integrate x^3.

	D	I
$+$	$\ln x$	x^3
$-$	$\frac{1}{x}$	$\frac{1}{4}x^4$

If we look at the final row, we can already integrate its product, so we stop.

	D	I
$+$	$\ln x$	x^3
$-$	$\frac{1}{x}$	$\frac{1}{4}x^4$

$$\begin{aligned}\int x^3 \ln x \, dx &= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 \, dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\end{aligned}$$

Example. Evaluate,

$$\int e^x \sin x \, dx$$

e^x and $\sin x$ are both easy to integrate and differentiate, so it doesn't really matter which way around we put them. Let's differentiate e^x and integrate $\sin x$.

	D	I
$+$	e^x	$\sin(x)$
$-$	e^x	$-\cos(x)$
$+$	e^x	$-\sin(x)$

We see that the final row is a copy of the first one (ignoring signs), so we can rewrite the integral as,

$$\begin{aligned}\int e^x \sin x \, dx &= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx \\ 2\int e^x \sin x \, dx &= -e^x \cos x + e^x \sin x \\ \int e^x \sin x \, dx &= \frac{1}{2}e^x \sin x - \frac{1}{2}e^x \cos x\end{aligned}$$

7.2 Weierstrass Substitution

The *Weierstrass substitution* is a change of variable that transforms rational functions of trigonometric functions into an ordinary rational function of a parameter, t.

Letting $t = \tan \frac{x}{2}$, we can transform the integral,

$$\int f(\sin x, \cos x) \, dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2}{1+t^2} \, dt$$

Geometrically, as x varies, the point $(\cos x, \sin x)$ travels across the unit circle at unit speed. In other words, it is a *unit speed parametrisation* (see MA134). The Weierstrass substitution is an alternative parametrisation of the unit circle such that the point $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ travels around the unit circle only once as t varies from $-\infty$ to ∞, starting and ending at $(-1,0)$. If you are familiar with projective geometry, this substitution can be viewed as the sterographic projection of the unit circle onto the y-axis from the point $(-1,0)$. This view can help you rederive various formulae on the fly, if required.

7.3 Reduction Formulae

A *reduction formula* allows you to write a recurrence relation for an integral in terms of related integrals with hopefully smaller exponents.

We do this by splitting up the exponent, substituting if needed, then integrating by parts.

Example.

$$\int \sin^n x \, dx$$

We wish to find a reduction formula for this integral. Start by setting,

$$\begin{aligned}
I_n &= \int \sin^n x \, dx \\
&= \int \sin^{n-1} x \sin x \, dx \\
&= -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x \cos^2 x \, dx \\
&= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx \\
&= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^2 x \, dx \\
&= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n \\
I_n + (n-1) I_n &= -\sin^{n-1} x \cos x + (n-1) I_{n-2} \\
I_n &= -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}
\end{aligned}$$

So now, if we're given, for example, $\int \sin^{100} x \, dx$, we can repeatedly apply the reduction formula until the power is low enough for us to evaluate the integral by hand.

7.4 Euler Substitution

If $f(a,b)$ is a rational function, then

$$\int f(x, \sqrt{ax^2 + bx + c}) \, dx$$

can be changed into the integral of a rational function using *Euler substitutions.*

If $a > 0$, solve $\sqrt{ax^2 + bx + c} = \pm x\sqrt{a} + t$ for x (the positive or negative sign can be chosen at will, depending on which is easier). The result will be a rational expression, that also allows us to write dx as a rational expression of t when we perform the substitution.

If $c > 0$, solve $\sqrt{ax^2 + bc + c} = xt \pm \sqrt{c}$ for x, and use the result as your substitution. Again, the positive and negative sign can be chosen at will.

If $ax^2 + bx + c$ has real roots, α,β, then we solve $\sqrt{a(x-\alpha)(x-\beta)} = (x-\alpha)t$ for x, which will again result in a rational expression.

7.5 Leibniz Integration Rule

$$\frac{d}{dx}\left(\int_a^b f(x,t) \, dt\right) = \int_a^b \frac{\partial}{\partial x} f(x,t) \, dt$$

There is a longer form for non-constant bounds of integration, but we will focus on the special case of constant bounds.

This theorem allows us to integrate functions we otherwise wouldn't be able to.

Example. Evaluate

$$\int_0^\infty \frac{\sin t}{t}\,dt$$

(The integrand is also known as the (unnormalised) *sinc function*, a function occuring often in signal processing contexts. This particular definite integral is the *Dirichlet integral*, and cannot be evaluated using standard methods.)

We begin by defining a function,

$$f(s) = \int_0^\infty e^{-st}\frac{\sin t}{t}\,dt$$

(If you are familiar with Laplace transforms, you might notice a similarity here. There is a much faster way of doing this with *Abel's theorem*, but that method will not be covered here, as it is beyond the scope of this document.)

We note that $f(0)$ is equal to the desired integral.

$$\begin{aligned}\frac{df}{ds} &= \frac{d}{ds}\int_0^\infty e^{-st}\frac{\sin t}{t}\,dt \\ &= \int_0^\infty \frac{\partial}{\partial s}e^{-st}\frac{\sin t}{t}\,dt \\ &= -\int_0^\infty e^{-st}\sin t\,dt\end{aligned}$$

You can alternatively use the complex definition of sine to perform this integral as an exercise.

$$\begin{aligned}&= -\frac{e^{-st}(\cos t + s\sin t)}{s^2+1}\Bigg|_{t=0}^{t=\infty} \\ &= -\frac{1}{s^2+1}\end{aligned}$$

Now, we integrate both sides with respect to s.

$$\begin{aligned}f(s) &= -\int \frac{1}{s^2+1}\,ds \\ &= -\arctan s + C \\ -\arctan s + C &= \int_0^\infty e^{-st}\frac{\sin t}{t}\,dt\end{aligned}$$

Here, we can try some values of s to get some information about C. $s = 0$ doesn't work, because we just get the original problem back. Let's see what happens as $s \to \infty$.

$$\begin{aligned}\lim_{s\to\infty} -\arctan s + C &= \lim_{s\to\infty}\int_0^\infty e^{-st}\frac{\sin t}{t}\,dt \\ \lim_{s\to\infty} -\arctan s + C &= \lim_{s\to\infty}\int_0^\infty \frac{\sin t}{te^{st}}\,dt \\ -\frac{\pi}{2} + C &= \int_0^\infty 0\,dt \\ -\frac{\pi}{2} + C &= 0\end{aligned}$$

$$C = \frac{\pi}{2}$$
$$f(s) = \frac{\pi}{2} - \arctan s$$
$$f(0) = \frac{\pi}{2} - 0$$
$$\int_0^\infty e^{-0t} \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$
$$\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$

7.6 Non-Elementary Integrals

If you somehow end up with one of these when constructing a differential equation for a question, you've probably done something wrong earlier.

The following is a non-exhaustive list of integrals that you will not be able to evaluate.

$$\int \sqrt{1+x^n}\, dx, \quad n \in \mathbb{N}, n \geq 3 \qquad \int \sin(\sin x)\, dx \qquad \int e^{e^x}\, dx$$
$$\int \sqrt{1-x^n}\, dx, \quad n \in \mathbb{N}, n \geq 3 \qquad \int \arcsin(\arcsin x)\, dx \qquad \int e^{x^2}\, dx$$
$$\int x^x\, dx \qquad \int \sin(x^2)\, dx \qquad \int e^{-x^2}\, dx$$
$$\int x^{-x}\, dx \qquad \int \cos(x^2)\, dx \qquad \int \frac{e^x}{x}\, dx$$
$$\int \frac{1}{\ln x}\, dx \qquad \int \frac{\sin x}{x}\, dx \qquad \int \frac{e^{-x}}{x}\, dx$$
$$\int \frac{x^n}{e^x - 1}\, dx \quad n \in \mathbb{N} \qquad \int \ln(\ln x)\, dx \qquad \int x^{c-1} e^{-x}\, dx, \quad c \notin \mathbb{N}$$

While you don't have to memorise all of these, it's good to be able to recognise when you have an integral you can't evaluate, so you can go back and check your previous working, rather than wasting time on the integral.